DAMTP-HEP-94/89

# A method to determine the operator content of perturbed conformal field theories

**Anni Koubek***

*Department of Applied Mathematics and Theoretical Physics,*
*Silver Street, CB3 9EW Cambridge, UK*
*E-mail: a.koubek@amtp.cam.ac.uk*

**Abstract**

A method to determine the full structure of the space of local operators of massive integrable field theories, based on the form factor bootstrap approach is presented. This method is applied to the integrable perturbations of the Ising conformal point. It is found that the content of local operators can be expressed in terms of fermionic sum representations of the characters $\tilde{\chi}(q)$ of the Virasoro irreducible representations of the minimal model $\mathcal{M}_{3,4}$. The space of operators factorises into chiral components as $Z = \sum \tilde{\chi}(q)\tilde{\chi}(\bar{q})$, but with the relation $\bar{q} = q^{-1}$.

October 1994
revised December 1994

*Work supported by PPARC grant no. GR/J20661

# 1 Introduction

In this letter we shall present a method in order to determine the operator content of an integrable massive field theory. It is based on the form factor bootstrap approach [1, 2], which provides a tool to classify *all* local operators of the theory under consideration. This is possible because the form factor equations are valid for every local operator of a theory. Therefore determining the whole solution space of these equations is equivalent to determine the content of local operators of the theory.

As an example, we shall apply this method mainly to the integrable perturbations of the critical Ising model. The thermal perturbation can be described in terms of free massive Majorana fermions, or equally in terms of a bosonic particle which interacts through an $S$-matrix $S = -1$. This theory consists of two sectors. One contains the monomials in the fermion fields, while the other sector contains fields which aquire a phase $e^{i\pi}$ when moved around the fermions. The basic fields in this second sector are the order and disorder fields $\sigma(x)$ and $\mu(x)$.

The magnetic perturbation is described by a massive field theory containing 8 particles [3], and is intrinsically related to the $E_8$ algebra. There are no internal symmetries in this theory and all operators are expected to be mutually local.

Let us summarise the description of an integrable theory in the form factor approach. We parametrise the momenta of the asymptotic states in terms of the mass $m$ of the particles and the rapidity variables $\beta_i$

$$p_i^0 = m \cosh\beta_i \ , \ p_i^1 = m \sinh\beta_i \ .$$

Form factors are matrix elements of a local operator $\mathcal{O}$ between the vacuum and the set of asymptotic states,

$$\mathcal{F}_n^{\mathcal{O}}(\beta_1, \beta_2, \ldots, \beta_n) = \langle 0 \mid \mathcal{O}(0,0) \mid Z(\beta_1), Z(\beta_2), \ldots, Z(\beta_n)\rangle_{in} \quad . \tag{1}$$

Their knowledge determines the correlation functions which can be expressed as

$$\langle \mathcal{O}(z,\bar{z})\, \mathcal{O}(0)\rangle = \tag{2}$$

$$\left(\frac{\bar{z}}{z}\right)^s \sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_n}{n!(2\pi)^n} |F_n^{\mathcal{O}}(\beta_1, \beta_2, \ldots, \beta_n)|^2 e^{-mr\sum_i \cosh\beta_i} \quad ,$$

where $r$ denotes the radial distance in the Euclidean space, *i.e.* $r = \sqrt{x_0^2 + x_1^2}$, and $s$ is the spin of the operator $\mathcal{O}$.

For systems with scalar particles, the functional equations known as the *form-factor axioms* are [1, 2, 4]:

| | |
|---|---|
| $\mathcal{F}^{\mathcal{O}}_{\epsilon_1\ldots\epsilon_i\epsilon_{i+1}\ldots\epsilon_n}(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots\beta_n) =$ <br> $S_{\epsilon_i\epsilon_{i+1}}(\beta_i-\beta_{i+1})\,\mathcal{F}^{\mathcal{O}}_{\epsilon_1\ldots\epsilon_{i+1}\epsilon_i\ldots\epsilon_n}(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots\beta_n)$ | (F1) |
| $\mathcal{F}^{\mathcal{O}}_{\epsilon_1\epsilon_2\ldots\epsilon_n}(\beta_1+2\pi i,\beta_2,\ldots,\beta_n) = e^{2i\pi(s_{\epsilon_n}+w_{\epsilon_n}+\sum w_{\epsilon_i\epsilon_n})}\mathcal{F}^{\mathcal{O}}_{\epsilon_2\ldots\epsilon_n\epsilon_1}(\beta_2,\ldots,\beta_n,\beta_1)$ <br> $s_{\epsilon_n}$ denotes the spin of the particle $\epsilon_n$; $w_\epsilon$ is the mutual locality index between <br> $\mathcal{O}$ and $Z_\epsilon$; $w_{\epsilon_i\epsilon_n}$ denotes the exchange properties of $Z_{\epsilon_i}$ and $Z_{\epsilon_n}$. | (F2) |
| $\mathcal{F}^{\mathcal{O}}_{\epsilon_1\ldots\epsilon_n}(\beta_1+\Lambda,\ldots,\beta_n+\Lambda) = e^{s\Lambda}\mathcal{F}^{\mathcal{O}}_{\epsilon_1\ldots\epsilon_n}(\beta_1,\ldots,\beta_n)$ | (F3) |
| $\mathcal{F}_{\epsilon_1,\ldots,\epsilon_n}(\beta_1+\Lambda,\ldots,\beta_i+\Lambda,\beta_{i+1},\ldots,\beta_n) = O(e^{S_n^i \vert\Lambda\vert}) \quad \text{for} \quad \vert\Lambda\vert \sim \infty$ <br> with $\max(S_n^i) < \infty$ | (F4) |
| $-i\lim_{\beta'\to\beta}(\beta'-\beta)\mathcal{F}^{\mathcal{O}}_{\epsilon\epsilon\epsilon_1\ldots\epsilon_n}(\beta'+i\pi,\beta,\beta_1,\ldots,\beta_n) =$ <br> $\left(1-e^{2i\pi(w_\epsilon+\sum w_{\epsilon_i\epsilon})}\prod_{i=1}^n S_{\epsilon\epsilon_i}(\beta-\beta_i)\right)\mathcal{F}^{\mathcal{O}}_{\epsilon_1\ldots\epsilon_n}(\beta_1,\ldots,\beta_n)$ | (F5) |
| If the two-particle scattering amplitude exhibits a pole with the residue <br> $-i\lim_{\beta'\to iu^{\epsilon_k}_{\epsilon_i\epsilon_j}}(\beta-iu^{\epsilon_k}_{\epsilon_i\epsilon_j})S_{\epsilon_i\epsilon_j}(\beta) = (\Gamma^{\epsilon_k}_{\epsilon_i\epsilon_j})^2$ , then <br> $-i\lim_{\beta'\to\beta}(\beta'-\beta)\mathcal{F}^{\mathcal{O}}_{\epsilon_1\ldots\epsilon_i\epsilon_j\ldots\epsilon_n}(\beta_1,\ldots,\beta'+i\bar{u}^{\epsilon_j}_{\epsilon_i\epsilon_k},\beta-i\bar{u}^{\epsilon_i}_{\epsilon_j\epsilon_k},\ldots,\beta_{n-1}) =$ <br> $= \Gamma^{\epsilon_k}_{\epsilon_i\epsilon_j}\mathcal{F}^{\mathcal{O}}_{\epsilon_1\ldots\epsilon_k\ldots\epsilon_n}(\beta_1,\ldots,\beta,\ldots,\beta_{n-1})$ | (F6) |

Further we will use the following notation: We introduce the character $\chi$ of the conformal minimal model as

$$\chi^{(p,q)}_{r,s} = q^{(h-c/24)}\tilde{\chi}^{(p,q)}_{r,s} \quad , \tag{3}$$

where $c$ and $h$ denote the central charge and the conformal dimension of the primary op-

erator respectively. Therefore $\tilde{\chi}$ encodes the degeneracies in the Virasoro representation [9],

$$\tilde{\chi}^{p,q}_{r,s} = \frac{1}{(q)_\infty} \sum_{\mu=-\infty}^{\infty} \left( q^{\mu(\mu pq + rq - sp)} - q^{(\mu p + r)(\mu q + s)} \right) \quad . \tag{4}$$

Further, define $P(m,n)$ to be the number of partitions of $n$ into numbers whose value do not exceed $m$. These partitions are generated by

$$\frac{1}{(q)_m} = \sum_{n=0}^{\infty} P(m,n) q^n \quad , \tag{5}$$

where

$$(q)_m \equiv \prod_{i=1}^{m} (1 - q^i) \quad . \tag{6}$$

# 2 The thermal perturbation of the Ising model

The theory contains only one particle, and the form factors can be parametrised as [4, 5, 6]

$$\mathcal{F}_n(\beta_1 \ldots \beta_n) = Q_n(x_1, \ldots, x_n) H_n(\sigma_n)^{\frac{1}{2}\delta_{w,0}} \frac{1}{(\sigma_n)^N} \prod_{i<j}^{n} \tanh \frac{(\beta_i - \beta_j)}{2} \quad , \tag{7}$$

where $\sigma_i$ denote the elementary symmetric polynomials [7], $H_n$ are normalisation constants and $x_i = e^{\beta_i}$. This parametrisation reduces the recursion relation (1) to

$$\begin{aligned} Q_{n+2}(-x, x, x_1, \ldots, x_n) &= x^{2N} Q_n(x_1, \ldots, x_n) \ , \quad w = \tfrac{1}{2} \quad , \\ Q_{n+2}(-x, x, x_1, \ldots, x_n) &= 0 \ , \qquad\qquad\qquad\qquad\quad w = 0 \quad . \end{aligned} \tag{8}$$

The integer $N$ in the parametrisation (7) introduces a grading into the space of operators. Increasing $N$ augments the divergence of the corresponding form factor, if the spin $s$ of the operator $\mathcal{O}$ is kept fixed. Therefore, in analogy to CFT, one can define as the chiral operators those with the mildest ultraviolet behaviour, $N = 0$ [5]. We have therefore two discrete parameters in the massive theory, namely the integer $N$ and the spin $s$. The spin $s$ can be related to the dimensions $(h, \bar{h})$ of the conformal fields which we obtain taking the UV limit as usual through $s = h - \bar{h}$. The parameter $N$ has no obvious interpretation in the conformal limit. Its role will be clarified later on.

## 2.1 Operators in the sector $w = \frac{1}{2}$

The sector $w = \frac{1}{2}$ contains the order and the disorder fields $\sigma$ and $\mu$, and has been investigated in [4, 5]. The form-factors of both operators have been calculated and are given by $N = 0$ and $Q_n = const.$ in (7); further, in [5] the descendent operators of $\sigma(x)$ have been determined.

The method we want to describe does in general not resolve the recursion relations but rather counts the number of linearly independent solutions iteratively. Namely we use the fact that the number of solutions of the recursion relations at level $n$, *i.e.* for the form factor $\mathcal{F}_n$, is given by the number of solutions at level $n-2$ plus the dimension of the kernel of the recursion relations (8). This amounts in a simple counting procedure which only involves the comparison of the degree of the polynomials $Q_n$ and that of the kernel. This has the advantage that one finds *all* solutions to the form factor equations, and not only those which can be considered as descendent operators of some non-trivial primary field. In the case of the Ising model the kernel of the recursion relation (8) is given by

$$\mathcal{K}_n = \prod_{i<j}^{n} (x_i + x_j) \quad . \tag{9}$$

The total degree of this function is $deg(\mathcal{K}_n) = \frac{1}{2}n(n-1)$. A Kernel solution is consistent for spin $s$ only if $deg(Q_n) \geq deg(\mathcal{K}_n)$, that is, $Q_n$ can be written as $Q_n = R(\sigma_i)\mathcal{K}_n$, where $R(\sigma_i)$ is an arbitrary combination of elementary symmetric polynomials of degree $deg(Q_n) - deg(\mathcal{K}_n)$.

Using this fact and the generating function (6), one finds that the number of solutions at general spin $s$, is generated by

$$F_0 \equiv \sum_{m,\, odd} \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} = \tilde{\chi}_{1,2}^{(3,4)} \quad , \tag{10}$$

where the exponents in the nominator corresponds to the spin value, where a new kernel solution enters. Note that this formalism leads to fermionic sum expressions [10] of the character.

Similarly one can analyse the even form-factors, being related to the disorder field $\mu$. The counting procedure can be carried out analogously, and one finds that the number

of chiral operators for spin $s$ is generated by

$$G_0 \equiv \sum_{m,\, even} \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} = \tilde{\chi}_{1,2}^{(3,4)} \quad . \tag{11}$$

As it is well known, both order and disorder fields are related to the conformal operator $\phi_{1,2}$.

As a next step we investigate solutions corresponding to higher $N$. Increasing $N$ we can gradually build up the whole space of operators in the massive theory. The number of operators at level $N$ is generated by

$$F_N \equiv \sum_{m=1,odd}^{\infty} \frac{q^{\frac{1}{2}m(m-1)-Nm}}{(q)_m}$$

in the odd sector, and by

$$G_N \equiv \sum_{m=0,even}^{\infty} \frac{q^{\frac{1}{2}m(m-1)-Nm}}{(q)_m}$$

in the even sector. These functions satisfy the recursion relations

$$F_N = F_{N-1} - \bar{q}^N G_{N-1} \quad , \qquad G_N = G_{N-1} + \bar{q}^N F_{N-1} \quad , \tag{12}$$

with the initial conditions $G_0 = F_0 = \tilde{\chi}_{1,2}(q)$, which can easily be solved to give $F_N = G_N = \prod_{k=1}^{N}(1+\bar{q}^k)F_0$. The whole space of states is generated by

$$\lim_{N\to\infty} F_N = \tilde{\chi}_{1,2}(\bar{q})\tilde{\chi}_{1,2}(q) \quad , \tag{13}$$

and similarly for $G_\infty$.

Let us summarise the key features of the method described, which generalise also to other integrable massive field theories [15]:

- The content of chiral operators is given by fermionic sum expressions of the characters of the corresponding Virasoro irreducible representation (or of sums of the characters).

- The whole space of operators is constructed by taking the limit $N \to \infty$. It can be formally decomposed into conformal characters as $\sum \tilde{\chi}(q)\tilde{\chi}(\bar{q})$ where the variables $q$ and $\bar{q}$ are *not* independent as in CFT but $\bar{q} = q^{-1}$.

- Finally, one obtains for finite $N$ some 'finitized' expressions for the Virasoro characters. In this approach they appear naturally in the counting procedure of states. Note that the same expressions turn up in the study of corner transfer matrices where the finitization is related to the fact that one has a discrete system of finite size (see *e.g.* [11] for the thermal perturbation of the Ising model). It would be interesting to understand whether there is a deeper reason for this relation.

## 2.2 Operators in the sector $w = 0$

For the sector $w = 0$ the kinematical recursion relation maps the form factors onto zero, $\mathcal{F}_n(\beta + i\pi, \beta, \ldots) = 0$. It follows that form factors with different particle number $n$ are not linked, and therefore *any* kernel solution will represent an acceptable form factor from the point of view of the form factor equations.

We start our analysis with the space related to the even form factors. The fundamental operator is the energy density, whose form factor is given by $\mathcal{F}_2 = \sinh \frac{\beta}{2}$. Comparing this with the parametrisation (7), we find that in this sector the primary operator corresponds to a solution with $N = 1$.

Using the counting method as before, we find that the solutions at level $N$ are generated by

$$f_N \equiv \sum_{n,even} \frac{q^{\frac{n^2}{2} - Nn}}{(q)_n} \quad . \tag{14}$$

Expressing the lowest terms in Virasoro characters, one finds that $f_0 = \tilde{\chi}_{1,2}(q)$ and $f_1$, which we conjectured to generate the chiral operators, is given by $f_1 = \tilde{\chi}_{1,1}(q) + \tilde{\chi}_{1,3}(q)$, as expected.

Since the form factors are not linked by the recursive equations for every operator in this sector there is only one contribution in the sum in (2), and the critical exponents can be calculated explicitly. One can show that all operators corresponding to solutions $Q_n$ in the space $N = 0$, are operators which will scale in the ultraviolet limit as elements of the identity module, while the operators in the $N = 1$ space form the space of descendents of the primary field $\phi_{1,3}$ [†].

[†] The operators of the space $N = 0$ appear in the space $N = 1$ as solutions $Q_n \sim \sigma_n$.

In order to find the decomposition of this expression into characters let us introduce

$$g_N \equiv \sum_{n,odd} \frac{q^{\frac{n^2-1}{2}-Nn}}{(q)_n} \quad . \tag{15}$$

These generating functions satisfy the recursion identities

$$\begin{aligned} f_N &= f_{N-1} + \bar{q}^{N-1} g_{N-1} \quad , \\ g_N &= g_{N-1} + \bar{q}^{N} f_{N-1} \quad , \end{aligned} \tag{16}$$

with initial conditions $f_0 = \tilde{\chi}_{1,1}$ and $g_0 = \tilde{\chi}_{1,3}$. Their solution is given by [13, 14]

$$\begin{aligned} f_N &= D_N^{(1)}(\bar{q})\tilde{\chi}_{1,1} + D_N^{(2)}(\bar{q})\tilde{\chi}_{1,3} \\ g_N &= \bar{q} D_N^{(2)}(\bar{q})\tilde{\chi}_{1,1} + D_N^{(1)}(\bar{q})\tilde{\chi}_{1,3} \end{aligned} \tag{17}$$

where

$$\begin{aligned} D_N^{(1)}(\bar{q}) &= \sum_{\mu=-\infty}^{\infty} \bar{q}^{\mu(12\mu+1)} \begin{bmatrix} 2N \\ N-4\mu \end{bmatrix} - \bar{q}^{(3\mu+1)(4\mu+1)} \begin{bmatrix} 2N \\ N-1-4\mu \end{bmatrix} \quad , \\ D_N^{(2)}(\bar{q}) &= \sum_{\mu=-\infty}^{\infty} \bar{q}^{\mu(12\mu+1)} \begin{bmatrix} 2N \\ N-1-4\mu \end{bmatrix} - \bar{q}^{(3\mu+1)(4\mu+2)} \begin{bmatrix} 2N \\ N-2-4\mu \end{bmatrix} \quad , \end{aligned} \tag{18}$$

and $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(q)_a}{(q)_{a-b}(q)_b}$ denote the Gaussian polynomials. These expressions satisfy

$$D_N^{(1)}(\bar{q}) \overset{N\to\infty}{\longrightarrow} \tilde{\chi}_{1,1}(\bar{q}) \quad , \quad D_N^{(2)}(\bar{q}) \overset{N\to\infty}{\longrightarrow} \tilde{\chi}_{1,3}(\bar{q}) \quad , \tag{19}$$

which means that for $N \to \infty$ the space of local operators in this sector can be written as

$$f_\infty = \tilde{\chi}_{1,1}(\bar{q})\tilde{\chi}_{1,1}(q) + \tilde{\chi}_{1,3}(\bar{q})\tilde{\chi}_{1,3}(q) \quad . \tag{20}$$

Finally we analyse the odd form factors in the sector $w = 0$. From the parametrisation (7) we find that form-factors of the operators with the mildest ultraviolet behaviour ($N = 0$) will have half integer spin. We choose as our basic operator the fermion, whose form factor is given by $\mathcal{F}_1 = \sigma_1^{\frac{1}{2}} = e^{\frac{\beta}{2}}$. Carrying out a similar analysis as for the even form factors one finds that the generating function for the chiral operators is given by $g_0 = \tilde{\chi}_{1,3}$, where $G_N$ is given in (15). One can easily check that the ultraviolet dimensions are those of the corresponding conformal operators.

The whole space of operators in this sector is generated by

$$g_\infty = \tilde{\chi}_{1,1}(\bar{q})\tilde{\chi}_{1,3}(q) + \bar{q}\tilde{\chi}_{1,3}(\bar{q})\tilde{\chi}_{1,1}(q) \quad . \tag{21}$$

The space of operators has the same structure as the space of descendent operators of the conformal operators $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

Summarising, we found the descendent spaces of the fermions, the energy density and the identity operator in this sector. All are expressed in terms of the characters of the critical Ising model. It is interesting to note that the counting method selects automatically a base in the space of operators which is isomorphic to the corresponding Virasoro irreducible representation.

## 3 The magnetic perturbation

The perturbation of the critical Ising model by the operator $\phi_{1,2}$ couples the model to an external magnetic field at critical temperature. This system develops a finite correlation length and hence is massive. The on-shell structure of the theory has been determined in [3], and contains 8 scalar massive particles. The $S$-matrix of the fundamental particle is given by

$$S_{11}(\beta_1 - \beta_2) = f_{2/3}(\beta_1 - \beta_2) f_{2/5}(\beta_1 - \beta_2) f_{1/15}(\beta_1 - \beta_2)$$

with

$$f_\alpha(\beta) \equiv \frac{\tanh \frac{1}{2}(\beta + i\alpha\pi)}{\tanh \frac{1}{2}(\beta - i\alpha\pi)} \quad . \tag{22}$$

All other $S$-matrix elements can be calculated by using the bootstrap equations. Because of the lack of space we are not able to give a detailed account of our investigations. We will present the major results, in order to show how the counting method can be applied also to relatively complicated systems, and give a complete presentation elsewhere [15].

We start our analysis by applying the counting method to the fundamental particle and include the others by using the bound state equation (F6). In order to parametrise the form-factor conveniently, we introduce

$$\zeta(\theta, \alpha) \equiv \exp\left\{ \int_0^\infty \frac{dx}{x} \frac{\cosh x(\frac{1}{2} - \alpha)}{\cosh \frac{x}{2}} \frac{\sin^2 \frac{x(i\pi - \beta)}{2\pi}}{\sinh x} \right\} \quad , \tag{23}$$

and

$$< \alpha > \equiv \sinh \frac{1}{2}(\beta + i\pi\alpha) \sinh \frac{1}{2}(\beta - i\pi\alpha) \quad .$$

The form-factors corresponding to $n_1$ particles of type 1 can then be parameterised as

$$\begin{gathered}\mathcal{F}_{n_1}(\beta_1, \dots, \beta_{n_1}) = \\ H_{n_1} Q_{n_1}(x_1, \dots, x_n) \frac{1}{\sigma_{n_1}^{N_1}} \prod_{i<j}^{n_1} \frac{\zeta(\beta_{ij}, \frac{2}{3})\zeta(\beta_{ij}, \frac{2}{5})\zeta(\beta_{ij}, \frac{1}{15})}{(x_i x_j)^{\frac{3}{2}} < 1 >^{\frac{1}{2}} < \frac{2}{3} >< \frac{2}{5} >< \frac{1}{15} >}\end{gathered} \tag{24}$$

where $\beta_{ij} = \beta_i - \beta_j$. Note that this parametrisation is an analytic continuation of the parametrisation of the form-factors of the particle of the Bullogh-Dodd model [16]. The degree of the kernel of the recursion relations is given by $deg(\mathcal{K}_{n_1}) = \frac{7}{2} n_1(n_1 - 1)$, while the degree of $Q_{n_1}$ is $deg(Q_{n_1}) = \frac{3}{2} n_1(n_1 - 1)$, which leads to the $q$-sum expression

$$\mathcal{T} = \sum_{m_1} \frac{q^{2m_1(m_1-1)}}{(q)_{m_1}} \quad . \tag{25}$$

In order to determine the contributions to this $q$-sum expression arising from particle 2, we apply the bound state equation to (24). For the form factors with $n_1$ particles 1, and $n_2$ particles 2 we find

$$\begin{aligned}\mathcal{F}_{1,\dots,1,2\dots,2} = H_{n_1,n_2} & \frac{1}{(\sigma_{n_1}(x_1, \dots, x_{n_1}))^{N_1} (\sigma_{n_2}(x_{n_1+1}, \dots, x_{n_1+n_2}))^{N_2}} \times \\ & \times Q_{n_1,n_2}(x_1, \dots, x_{n_1}; x_{n_1+1}, \dots, x_{n_1+n_2}) \times \\ & \times \prod_{i<j}^{n_1} \frac{\zeta(\beta_{ij}, \frac{2}{3})\zeta(\beta_{ij}, \frac{2}{5})\zeta(\beta_{ij}, \frac{1}{15})}{(x_i x_j)^{\frac{3}{2}} < 1 >^{\frac{1}{2}} < \frac{2}{3} >< \frac{2}{5} >< \frac{1}{15} >} \times \\ & \times \prod_{i=1}^{n_1} \prod_{i=1}^{n_2} \frac{1}{(x_i x_j)^3} \frac{\zeta(\frac{7}{15})\zeta(\frac{3}{5})\zeta(\frac{1}{5})\zeta(\frac{4}{15})}{< \frac{4}{5} >< \frac{3}{5} >< \frac{4}{15} >< \frac{13}{15} >< \frac{7}{15} >} \times \\ & \times \prod_{i<j}^{n_2} \frac{1}{(x_i x_j)^6} \frac{< 0 > \zeta(\frac{2}{3})\zeta(\frac{4}{15})\zeta(\frac{1}{5})\zeta(\frac{2}{5})^2\zeta(\frac{7}{15})\zeta(\frac{1}{15})}{< 1 >< \frac{2}{3} >< \frac{4}{5} >< \frac{7}{15} >< \frac{1}{15} >< \frac{4}{15} >< \frac{3}{5} >< \frac{2}{5} >< \frac{16}{15} >< \frac{2}{3} >} \quad .\end{aligned} \tag{26}$$

Comparing the dimension of the functions $Q_{12}$ and $Q_{22}$ with the dimension of the respective kernels, one finds that the exponent due to interaction of particles 1 and 2 is $2m_1^2 + 4m_2^2 + 4m_1 m_2 - 2m_1 - 4m_2$, modifying the expression (25) to

$$\mathcal{T} = \sum_{m_1,m_2} \frac{q^{\left\{ (m_1,m_2) \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} - 2m_1 - 4m_2 \right\}}}{(q)_{m_1}(q)_{m_2}} \quad . \tag{27}$$

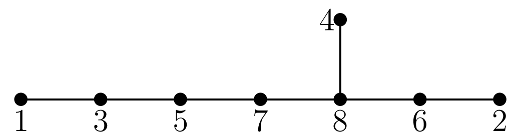


Figure 1: The $E_8$ Dynkin diagram, with the labelling of the nodes used in the definition of the generating function (28).

Notice that the pole structure in (26) is not minimal (in the sense that they exhibit poles also in non-physical locations). This is no obstacle for the counting method since only the *difference* between the degrees of the functions $Q$ and the kernels $\mathcal{K}$ enters.

Similarly it is possible to carry out the bootstrap for all particles. The corresponding expression of the form-factors becomes quite lengthly, and will be omitted. It will contain factors $1/\sigma_{n_i}^{N_i}$, $i = 1, \ldots, 8$, corresponding to each of the 8 particles. The generating functions for levels $N_1, \ldots, N_8$ can be found to be

$$\mathcal{T}_{N_1,N_2,\ldots,N_8} \equiv \mathcal{T}_N = \sum_M \frac{q^{MC^{-1}M - NC^{-1}M}}{(q)_{m_1} \ldots (q)_{m_8}} \tag{28}$$

where $M$ denotes the multi-index $M = m_1, \ldots m_8$ and $C^{-1}$ is the inverse Cartan matrix of the algebra $E_8$. The relevant labelling of the nodes of the $E_8$ Dynkin diagram is depicted in fig. 1. The linear terms in the $q$ sum expressions have been chosen in order to be consistent with the bootstrap.

The lowest of these functions is $\mathcal{T}_0 = \tilde{\chi}_{1,1}(q)$ [10, 17], corresponding to the most relevant operators, while the chiral operators are given by [10]

$$\mathcal{T}_{0,1,0,0,0,0,0,0} = \tilde{\chi}_{1,1}(q) + \tilde{\chi}_{1,2}(q) + \tilde{\chi}_{1,3}(q) \quad .$$

Similar higher functions can be expressed in terms of characters as for example

$$\begin{aligned}
\mathcal{T}_{0,0,1,0,0,0,0,0} &= \tilde{\chi}_{1,1}(q) + (1+\bar{q})\tilde{\chi}_{1,2}(q) + \tilde{\chi}_{1,3}(q) \\
\mathcal{T}_{0,0,0,1,0,0,0,0} &= \tilde{\chi}_{1,1}(q) + (1+\bar{q})\tilde{\chi}_{1,2}(q) + (1+\bar{q})\tilde{\chi}_{1,3}(q) \\
\mathcal{T}_{0,0,0,0,1,0,0,0} &= (1+\bar{q}^2)\tilde{\chi}_{1,1}(q) + (1+\bar{q}+\bar{q}^2)\tilde{\chi}_{1,2}(q) + (1+\bar{q}+\bar{q}^2)\tilde{\chi}_{1,3}(q) \quad .
\end{aligned} \tag{29}$$

These expressions correspond to the scalar partition function of the Ising model in its lowest orders in $\bar{q}$.

The generating functions (28) satisfy the recursion identities

$$\mathcal{T}_{N+\delta_{N_i}} = \mathcal{T}_{N-C\delta_{N_i}} + q^{C^{-1}N}\mathcal{T}_{N-\delta_{N_i}} \quad ,$$

which allow to reconstruct the space of operators for higher levels. We have carried out extensive numerical calculations which show that for $N \to \infty$ the functions $\mathcal{T}_N$ approach the scalar partition function

$$Z = \tilde{\chi}_{1,1}(q)\tilde{\chi}_{1,1}(\bar{q}) + \tilde{\chi}_{1,2}(q)\tilde{\chi}_{1,2}(\bar{q}) + \tilde{\chi}_{1,3}(q)\tilde{\chi}_{1,3}(\bar{q}) \tag{30}$$

as expected. It would be interesting to find exact solutions for these recursion relations, and determine their relation to the finitized characters for this model, which arise from carrying out the Bethe ansatz to the dilute $A_3$-model [17].

# 4 Conclusions

We presented a method in order to determine the full content of local operators of an integrable massive field theory. We have discussed its application to the integrable perturbations of the critical Ising model. It can be applied to any massive integrable theory, and in [15] we will discuss more examples explicitly.

The method consists of a simple counting procedure and leads to fermionic sum expressions of the characters, or of sums of the characters. Further, by increasing the possible divergence of the form-factors, the whole space of operators can be obtained. We found that this space can be formally factorised into characters of the corresponding Virasoro irreducible representation of the form $\sum \tilde{\chi}(q)\tilde{\chi}(\bar{q})$. This is a quite remarkable result, since the chiral sectors are not independent in the massive model. This fact is reflected by the relation $\bar{q} = q^{-1}$.

Though the structures of the conformal and massive theories are quite similar, there is one important difference. The grading in the space of conformal operators is introduced through the conformal weights of the operators, $h$ and $\bar{h}$, while in the massive model it is given through the spin $s$ and the parameters $N$. While the spin can be linked to

the conformal weights as $s = h - \bar{h}$ the parameters $N$ lack such a direct interpretation in terms of the CFT weights. It is our belief that understanding this connection of $N$ to the Virasoro structure in the ultraviolet limit, will give an indication of the algebraic structure determining the space of local operators in the massive theory.

# References


[1] B. Berg, M. Karowski, P. Weisz, *Phys. Rev.* **D19** (1979), 2477; M. Karowski, P. Weisz, *Nucl. Phys.* **B139** (1978), 445; M. Karowski, *Phys. Rep.* **49** (1979), 229;

[2] F.A. Smirnov, "Form–Factors in Completely Integrable Models of Quantum Field Theory", (World Scientific 1992); F. A. Smirnov, in *Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory*, Nankai Lectures on Mathematical Physics, World Scientific 1990;

[3] A.B. Zamolodchikov, in *Advanced Studies in Pure Mathematics* **19** (1989), 641; *Int. J. Mod. Phys.***A3** (1988), 743;

[4] V.P. Yurov and Al. B. Zamolodchikov, *Int. J. Mod. Phys.* **A6** (1991), 3419;

[5] J.L. Cardy and G. Mussardo, *Nucl. Phys.* **B340** (1990), 387;

[6] G. Mussardo, Lecture Notes of *"Summer School in High Energy Physics and Cosmology"* Trieste 1993;

[7] I.G. MacDonald, Symmetric Functions and Hall Polynomials (Clarendon Press, Oxford, 1979);

[8] P. Christe, *Int. J. Mod. Phys.* **A6** (1991), 5271;

[9] B.L. Feigin and D.B. Fuchs, *Funct. Anal. Appl.* **17** (1983), 241; A. Rocha-Caridi, in "Vertex Operators in Mathematics and Physics" eds. J. Lepowsky, S. Mandelstam and I.M. Singer (Springer 1985), 451;

[10] R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, *Phys. Lett.* **B304** (1993), 263; *Phys. Lett.* **B307** (1993), 68;

[11] R.J. Baxter, "Exactly solved models in statistical mechanics" (Academic Press 1982); J.L. Cardy, in "Fields, strings and critical phenomena" Proceedings Les Houches 1988, eds. E. Brezin and J. Zinn-Justin;

[12] I.S. Gradshteyn and I.M. Ryzhik "Table of Integrals, Series, and Products" (Academic Press 1965);

[13] G.E. Andrews, R.J. Baxter and P.J. Forrester, *J. Stat. Phys.* **35** (1984), 193;

[14] E. Melzer, *Int. J. Mod. Phys.* **A9** (1994), 1115;

[15] A.Koubek, in preparation;

[16] A. Fring, G. Mussardo and P. Simonetti, *Phys. Lett.* **B307** (1993), 83;

[17] S. O. Warnaar and P.A. Pearce, *A-D-E Polynomial and Rogers-Ramanujan Identities* hep-th/9411009; *Exceptional structure of the dilute $A_3$ model. $E_8$ and $E_7$ Rogers-Ramanujan identities* hep-th/9408136;